

ON HOLOMORPHIC DOMINATION, I

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ABSTRACT. Let X be a separable Banach space and $u: X \rightarrow \mathbb{R}$ locally upper bounded. We show that there are a Banach space Z and a holomorphic function $h: X \rightarrow Z$ with $u(x) < \|h(x)\|$ for $x \in X$. As a consequence we find that the sheaf cohomology group $H^q(X, \mathcal{O})$ vanishes if X has the bounded approximation property (i.e., X is a direct summand of a Banach space with a Schauder basis), \mathcal{O} is the sheaf of germs of holomorphic functions on X , and $q \geq 1$. As another consequence we prove that if f is a C^1 -smooth $\bar{\partial}$ -closed $(0, 1)$ -form on the space $X = L_1[0, 1]$ of summable functions, then there is a C^1 -smooth function u on X with $\bar{\partial}u = f$ on X .

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1. INTRODUCTION.

The ideas of plurisubharmonic domination and holomorphic domination along with some of their applications appeared in [L3] by Lempert. Following him we say that *plurisubharmonic domination* is possible on a complex Banach manifold M if for every locally upper bounded $u: M \rightarrow \mathbb{R}$ there is a continuous plurisubharmonic $\psi: M \rightarrow \mathbb{R}$ with $u(x) < \psi(x)$ for all $x \in M$. If ψ can be taken in the form $\psi(x) = \|h(x)\|$ for $x \in M$, where $h: X \rightarrow Z$ is a holomorphic function to a Banach space Z , then we say that *holomorphic domination* is possible in M .

One tool to achieve holomorphic domination is the following Runge approximation property of a Banach space X .

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² To my younger brother.

Hypothesis 1.1. [L3, Hypothesis 1.5] *There is a constant $0 < \mu < 1$ such that if Z is any Banach space, $\varepsilon > 0$, and $f: B_X \rightarrow Z$ is holomorphic on the open unit ball B_X of X , then there is a holomorphic function $g: X \rightarrow Z$ with $\|f(x) - g(x)\| < \varepsilon$ for $\|x\| < \mu$.*

Lempert and Meylan proved the following theorem involving the above.

Theorem 1.2. (a) (Lempert, [L2]) *If X is a Banach space with an unconditional basis, then Hypothesis 1.1 above holds for X .*

(b) (Meylan, [M]) *If X is a Banach space with an unconditional finite dimensional Schauder decomposition, then Hypothesis 1.1 holds for X .*

(c) (Lempert, [L3]) *If X is a Banach space with a Schauder basis (or a direct summand of one) and Hypothesis 1.1 holds for X , then holomorphic domination is possible in every pseudoconvex open subset of X .*

Our main goal in this paper is to find a route to holomorphic domination that bypasses Hypothesis 1.1 above. Our main results are Theorems 1.3, 1.4, 1.5, and 6.1 below.

Theorem 1.3. *If X is a separable Banach space, then holomorphic domination is possible (a) in X , and (b) in every convex open $\Omega \subset X$.*

As a consequence of Theorem 1.3 we get cohomology vanishing as follows.

Theorem 1.4. *Let X be a Banach space with the bounded approximation property, $\Omega \subset X$ pseudoconvex open, $M \subset \Omega$ a closed split complex Banach submanifold of Ω , $S \rightarrow M$ a cohesive sheaf, $E \rightarrow \Omega$ a holomorphic Banach vector bundle, and $I \rightarrow \Omega$ the sheaf of germs of holomorphic sections of E over Ω that vanish on M . If plurisubharmonic domination is possible in Ω (which is guaranteed by Theorem 1.3 if $\Omega \subset X$ is convex open), then the following hold.*

- (a) *The cohesive sheaf $S \rightarrow M$ admits a complete resolution over M .*
- (b) *The sheaf cohomology group $H^q(M, S)$ vanishes for all $q \geq 1$.*
- (c) *The sheaf I is cohesive over Ω , $H^q(\Omega, I) = 0$ for $q \geq 1$, and any holomorphic section $f \in \mathcal{O}(M, E)$ extends to a holomorphic section $F \in \mathcal{O}(\Omega, E)$ with $F(x) = f(x)$ for $x \in M$.*
- (d) *If $\Omega \subset X$ is convex open, then E is holomorphically trivial over Ω .*

As a consequence of Theorem 1.4 we get the following Theorem 1.5 on the $\bar{\partial}$ -equation.

Theorem 1.5. *Let X be an \mathcal{L}_1 -space with the bounded approximation property (e.g., $X = L_1[0, 1]$), $\Omega \subset X$ pseudoconvex open, $E \rightarrow \Omega$ a holomorphic Banach vector bundle, and $f \in C^1_{0,1}(\Omega, E)$ a C^1 -smooth $\bar{\partial}$ -closed $(0, 1)$ -form with values in E . If plurisubharmonic domination is possible in Ω (which*

is guaranteed by Theorem 1.3 if $\Omega \subset X$ is convex open), then there is a C^1 -smooth section $u \in C^1(\Omega, E)$ of E with $\bar{\partial}u = f$ over Ω .

Our strategy is to imitate the relevant parts of [L3] closely, but refrain from using Runge approximation for functions unbounded on balls. The reader is assumed to have a copy of [L3] along side this paper. In our §§ 2-4 we adopt without comment the notation of [L3, §§ 2-4].

2. BACKGROUND.

In this section we recall some material useful later. The paper [L3] uses a particular exhaustion $\Omega_N \langle \alpha \rangle$, $N \geq 1$, of any pseudoconvex open subset Ω of any Banach space X with a bimonotone Schauder basis, and there are numerous other sets used there to help out with the analysis of the said exhaustion. In our case all the sets involved will be convex open in X or in the span of finitely many of its basis vectors. The infinite dimensional ones among the sets that we need are all of the form $D \times B$, where D is a convex open set in the span of the first few basis vectors and B is a ball in the closed span of the rest of the basis vectors. As we shall need very little of the properties of the many sets discussed in [L3] we just help ourselves directly to the results there and skip any of their details (even their definitions) here.

In a Banach space X , put $B_X(x_0, r) = \{x \in X : \|x - x_0\| < r\}$ for the open ball of radius r centered at $x_0 \in X$, and write $B_X = B_X(0, 1)$ for the unit ball. Denote by $\mathcal{O}(M_1, M_2)$ the set of holomorphic functions $M_1 \rightarrow M_2$ from one complex Banach manifold M_1 to another M_2 .

Let X be a Banach space, $A \subset X$, and $u: A \rightarrow \mathbb{R}$. We say that u can be *dominated by entire functions with values in Banach spaces* on A if there are a Banach space Z and an entire holomorphic function $h \in \mathcal{O}(X, Z)$ with $u(x) \leq \|h(x)\|$ for all $x \in A$.

If T is any set, then denote by $\ell_\infty(T)$ the Banach space of bounded functions $f: T \rightarrow \mathbb{C}$ with the sup norm $\|f\| = \sup\{\|f(t)\| : t \in T\}$.

3. DOMINATION ON THE WHOLE SPACE.

In this section we show that if a function u can be dominated on every ball of a fixed radius, then u can be dominated on the whole space as well.

Let X be Banach space with a Schauder basis. Fix the norm and the Schauder basis of X so as to make a bimonotone Schauder basis of X . Fix $N \geq 1$ and write π for the Schauder projection onto the span of the first $N+1$ basis vectors, $\varrho = 1 - \pi$ for the complementary projection, and $Y = \varrho X$ for the complementary space.

Proposition 3.1. *If X is a Banach space with a bimonotone Schauder basis,*

$0 < R < \infty$, $u: X \rightarrow [1, \infty)$ is continuous, and u can be dominated by entire functions with values in Banach spaces on every ball $B_X(x_0, R)$ of radius R and centered at any $x_0 \in X$, then u can be dominated by entire functions with values in Banach spaces on X .

The proof of Proposition 3.1 will occupy us for a while.

Proposition 3.2. (Cf. [L3, Lemma 4.1]) *Let $A_2 \subset\subset A_3$ be relatively open bounded convex subsets of $\pi(X) \cong \mathbb{C}^{N+1}$, A_1 a compact convex subset of A_2 , and $0 < r_1 < r_2 < r_3 < \infty$ constants. If Z is a Banach space and $g \in \mathcal{O}(X, Z)$ is an entire function, then there are a Banach space W and an entire function $h \in \mathcal{O}(X, W)$ with*

- (i) $\|h(x)\|_W \leq 1$ for $x \in A_1[r_1]$ and
- (ii) $\|h(x)\|_W \geq \|g(x)\|_Z$ for $x \in A_3(r_3) \setminus A_2(r_2)$.

Proof. Consider the bounded convex sets H_1, H_2, H_3 in $\pi(X) \times \mathbb{C} \cong \mathbb{C}^{N+2}$ given by $H_1 = \{(s, \lambda) \in A_1 \times \mathbb{C} : |\lambda| \leq r_1\}$, $H_i = \{(s, \lambda) \in A_i \times \mathbb{C} : |\lambda| < r_i\}$ for $i = 2, 3$. Since H_1 is compact convex in \mathbb{C}^{N+2} there are a finite set J and polynomials $\varphi_j \in \mathcal{O}(\pi(X) \times \mathbb{C})$ for $j \in J$ such that $|\varphi_j(s, \lambda)| \leq \frac{1}{4}$ for $(s, \lambda) \in H_1$ and for every $(s, \lambda) \in H_3 \setminus H_2$ there is a $j \in J$ with $|\varphi_j(s, \lambda)| \geq 4$. Denote by $L = \overline{B_{Y^*}}$ the set of all linear functionals $l \in Y^*$ with $\|l\| \leq 1$, and by $V = \ell_\infty(L \times J)$. Define $\varphi \in \mathcal{O}(X, V)$ by $\varphi(x)(l, j) = \varphi_j(\pi x, l \varrho x)$ for $x \in X$, $l \in L$, and $j \in J$.

The rest of the proof of Proposition 3.2 is the same word for word as that of [L3, Lemma 4.1] starting with “Going back” near [L3, (4.1)].

Proposition 3.3. (Cf. [L3, Proposition 4.2]) *Let $0 < \mu < 1$, $N \geq 1$, and $2^4\beta < \alpha < 2^{-8}\mu$. If Z is a Banach space and $g \in \mathcal{O}(X, Z)$ is an entire function, then there are a Banach space W and an entire function $h \in \mathcal{O}(X, W)$ such that*

- (i) $\|h(x)\|_W \leq 1$ for $x \in \Omega_N\langle\beta\rangle$ and
- (ii) $\|h(x)\|_W \geq \|g(x)\|_Z$ for $x \in \Omega_{N+1}\langle\alpha\rangle \setminus \Omega_N\langle\alpha\rangle$.

Proof. In Proposition 3.3 the sets $\Omega_N\langle\beta\rangle$, etc, refer to those constructed in [L3, §3] for $\Omega = X$. Proposition 3.3 follows from Proposition 3.2 in the same way as [L3, Proposition 4.2] does from [L3, Lemma 4.1] only more simply.

Proof of Proposition 3.1. On replacing u by $u(Rx/2)$ we may assume that $R = 2$. Let $\Omega = X$, fix $0 < \mu < 1$ and $0 < \alpha < 2^{-8}\mu$. First, we construct a Banach space Z_N and an entire function $g_N \in \mathcal{O}(X, Z_N)$ for each $N \geq 1$. The set $A = \overline{\Omega_N\langle\alpha\rangle} \cap \pi_N(X)$ is compact and if $t \in A$, then $\Omega_N\langle\alpha\rangle \cap \pi_N^{-1}(t) \subset B_X(t, \alpha)$. Hence t has an open neighborhood $U \subset \pi_N(X)$ with $\Omega_N\langle\alpha\rangle \cap$

$\pi_N^{-1}(U) \subset B_X(t, 2\alpha)$. Therefore

$$(3.1) \quad \Omega_N \langle \alpha \rangle \subset \bigcup_{t \in T} B_X(t, 2\alpha)$$

for some finite $T \subset A$. Let $B_t = B_X(t, 2\alpha/\mu)$, the radius of which is less than 2. By our assumption that u can be dominated by entire functions with values in Banach spaces on $B_X(x_0, 2)$ for every $x_0 \in X$, there are a Banach space V_t and an entire function $f_t \in \mathcal{O}(X, V_t)$ with $u(x) \leq \|f_t(x)\|_{V_t}$ for $x \in B_t$, $t \in T$. Let Z_N be the ℓ_∞ -sum of the finitely many Banach spaces V_t for $t \in T$ and $g_N \in \mathcal{O}(X, Z_N)$ the map whose components are the f_t for $t \in T$. We see from (3.1) that $u(x) \leq \|g_N(x)\|_{Z_N}$ for $x \in \Omega_N \langle \alpha \rangle$.

The rest of the proof of Proposition 3.1 is the same as that of [L3, Proposition 2.1] starting with “In the second step” on page 368 there.

4. DOMINATION ON A BALL.

In this section we show that if a function u can be dominated on every ball of half the radius of a ball B and centered at any point of B , then u can be dominated on B itself.

Proposition 4.1. *If X is a Banach space with a bimonotone Schauder basis, $0 < R < \infty$, $u: X \rightarrow [1, \infty)$ is continuous, and u can be dominated by entire functions with values in Banach spaces on every ball $B_X(x_0, R/2)$ of radius $R/2$ and centered at any $x_0 \in B = B_X(y_0, R)$, then there is continuous function $\tilde{u}: X \rightarrow [1, \infty)$ such that $\tilde{u}(x) \leq u(x)$ for all $x \in X$, $\tilde{u}(x) = u(x)$ for $x \in B$, and \tilde{u} can be dominated by entire functions with values in Banach spaces on every ball $B_X(x_0, R/8)$ of radius $R/8$ centered at any $x_0 \in X$.*

Proof. Let $\chi: [0, \infty) \rightarrow [0, 1]$ be a cutoff function

$$\chi(t) = \begin{cases} 1 & 0 \leq t \leq R \\ 1 - \frac{4}{R}(t - R) & \text{if } R \leq t \leq \frac{5}{4}R, \\ 0 & t \geq \frac{5}{4}R \end{cases}$$

and define \tilde{u} by $\tilde{u}(x) = \chi(\|x - y_0\|)u(x) + 1 - \chi(\|x - y_0\|)$ for $x \in X$. As $\tilde{u}(x) - u(x) = (1 - \chi(\|x - y_0\|))(1 - u(x)) \leq 0$, being the product of a nonnegative number by a nonpositive number, we get that $\tilde{u}(x) \leq u(x)$ for all $x \in X$. Hence \tilde{u} can be dominated by entire functions with values in Banach spaces on any set on which u can.

If $x_0 \in X$ satisfies that $\|x_0 - y_0\| \geq \frac{11}{8}R$, then $B_X(x_0, \frac{1}{8}R)$ lies outside $B_X(y_0, \frac{5}{4}R)$ since the distance $\|x_0 - y_0\|$ of their centers exceeds the sum of their radii $\frac{5}{4}R + \frac{1}{8}R = \frac{11}{8}R$. Hence $\tilde{u} = 1$ on $B_X(x_0, \frac{1}{8}R)$, and so \tilde{u} can be dominated by entire functions with values in Banach spaces on $B_X(x_0, \frac{1}{8}R)$.

If $\|x_0 - y_0\| < R$, then $x_0 \in B_X(y_0, R)$ and $B_X(x_0, \frac{1}{8}R) \subset B_X(x_0, \frac{1}{2}R)$.

If $R \leq \|x_0 - y_0\| < \frac{11}{8}R$, then choose a value $0 < R' < R$ with $\|x_0 - y_0\| < \frac{11}{8}R'$, and let $z_0 = y_0 + R' \frac{x_0 - y_0}{\|x_0 - y_0\|}$. Then $\|z_0 - x_0\| = R' < R$ so $z_0 \in B_X(y_0, R)$ and we claim that $B_X(x_0, \frac{1}{8}R) \subset B_X(z_0, \frac{1}{2}R)$. To that end we must show that the distance $\|z_0 - x_0\|$ of the centers is less than the difference of the radii, i.e., $\|z_0 - x_0\| < \frac{1}{2}R - \frac{1}{8}R = \frac{3}{8}R$. Indeed, $\|z_0 - x_0\| = \|y_0 - x_0 + R' \frac{x_0 - y_0}{\|x_0 - y_0\|}\| = \|x_0 - y_0\| - R' < \frac{11}{8}R' - R' = \frac{3}{8}R' < \frac{3}{8}R$. The proof of Proposition 4.1 is complete.

Proposition 4.2. *If X is a Banach space with a bimonotone Schauder basis, $0 < R < \infty$, $u: X \rightarrow [1, \infty)$ is continuous, and u can be dominated by entire functions with values in Banach spaces on every ball $B_X(x_0, R/2)$ of radius $R/2$ centered at any $x_0 \in B = B_X(y_0, R)$, then u can be dominated by entire functions with values in Banach spaces on the ball B .*

Proof. Proposition 4.1 gives us a \tilde{u} that can be dominated by entire functions with values in Banach spaces on every ball of radius $R/8$ in X . Proposition 3.1 gives us a Banach space Z and an entire function $h \in \mathcal{O}(X, Z)$ with $\tilde{u}(x) \leq \|h(x)\|$ for all $x \in X$. As $u(x) = \tilde{u}(x) \leq \|h(x)\|$ for $x \in B$, the proof of Proposition 4.2 is complete.

5. PREPARATION.

This section is preparatory to the proofs of Theorems 1.3, 1.4, and 1.5.

Recall the following theorem of Pełczyński's.

Theorem 5.1. (Pełczyński, [P]) *A Banach space X has the bounded approximation property if and only if X is isomorphic to a direct summand of a Banach space Y with a Schauder basis, i.e., there are a Banach space Y with a Schauder basis and a direct decomposition $Y = Y_1 \oplus Y_2$ of Banach spaces such that $X \cong Y_1$.*

In most of our proofs we can avoid dealing with Banach spaces with the bounded approximation property, and only work with Banach spaces with a Schauder basis.

Proposition 5.2. *Let X be a Banach space with the bounded approximation property, and $\Omega \subset X$ pseudoconvex open. If plurisubharmonic domination is possible in Ω , then so is holomorphic domination.*

Proof. It is enough by Theorem 5.1 to prove this when X has a Schauder basis, in which case it follows from the argument of [L3], only more simply.

Proposition 5.3. *If M_0 is a closed complex Banach submanifold of a complex Banach manifold M , and holomorphic domination is possible on M , then*

holomorphic domination is possible on M_0 , too.

Proof. Let $u_0: M_0 \rightarrow \mathbb{R}$ be the locally upper bounded function to be dominated. Define $u: M \rightarrow \mathbb{R}$ by setting $u(x) = u_0(x)$ for $x \in M_0$ and $u(x) = 0$ otherwise. Clearly, u is locally upper bounded, M_0 being a closed subset of M . If Z is Banach space and $h \in \mathcal{O}(M, Z)$ dominates u on M , then the restriction h_0 of h to M_0 is holomorphic and dominates u_0 in M_0 . The proof of Proposition 5.3 is complete.

Proposition 5.4. *If M is a separable complex Banach manifold that is biholomorphic to a closed Banach submanifold of a Banach space X , then M can be embedded in a separable Banach space as a closed complex Banach submanifold.*

Proof. It is easy to see that the closed linear span of a separable subset of any Banach space is itself separable. It is a standard theorem that any separable Banach space is isomorphic to a closed linear subspace of the space $Y = C[0, 1]$ of continuous functions, and Y has a Schauder basis. Thus M is biholomorphic to a closed complex Banach submanifold of Y , completing the proof of Proposition 5.4.

Proposition 5.5. *Let X be a Banach space, and $\Omega \subset X$ open. If one of (a), (b), (c) below holds, then Ω is biholomorphic to a closed complex Banach submanifold M of a Banach space Y .*

(a) Ω is convex.

(b) *There is a direct decomposition $X = X_1 \oplus X_2$ of Banach spaces with $\dim_{\mathbb{C}}(X_1) < \infty$, and Ω is of the form $\Omega = \{(x_1, x_2) \in D \times X_2: \|x_2\| < R(x_1)\}$, where $D \subset X_1$ is pseudoconvex (relatively) open, $R: D \rightarrow (0, \infty)$ is continuous and $-\log R$ is plurisubharmonic on D .*

(c) Ω is of the form $\Omega = \{x \in \Omega': \|f(x)\| < 1\}$, where $\Omega' \subset X$ is open, the closure $\overline{\Omega} \subset \Omega'$, and $f \in \mathcal{O}(\Omega', Z_1)$ is holomorphic with values in a Banach space Z_1 .

Proof. In each case we define a Banach space Z and a holomorphic function $h \in \mathcal{O}(\Omega, Z)$ with $\liminf_{\Omega \ni x \rightarrow x_0} \|h(x)\| = \infty$ for each boundary point $x_0 \in \partial\Omega$. Then the graph $M \subset Y = X \times Z$ of h defined by $M = \{(x, z) \in \Omega \times Z: z = h(x)\}$ does the job.

(a) (See also [Pt1, Proposition 8.2].) Assume as we may that $0 \in \Omega$. Let $p: X \rightarrow \mathbb{R}$ be the Minkowski functional $p(x) = \inf\{\lambda > 0: \frac{x}{\lambda} \in \Omega\}$ of the convex open set Ω , and $K = \{\xi \in X^*: \operatorname{Re}(\xi x) \leq p(x) \text{ for all } x \in X\}$. Then $K \neq \emptyset$ is a convex subset of the dual space X^* of X . We endow K with the weak star topology, in which K is compact.

Let $Z = C([0, 2\pi] \times K, \mathbb{C})$ be the usual Banach space with the sup norm,

$g(t) = 1/(1-t)$ for $t \in B_{\mathbb{C}}$, and define for $x \in \Omega$ a function $h(x) \in Z$ by $h(x)(\theta, \xi) = g(e^{i\theta} e^{\xi x - 1})$. Then $\|h(x)\| = \sup\{|h(x)(\theta, \xi)| : \theta \in [0, 2\pi], \xi \in K\} \leq \sup_{\theta, \xi} g(|e^{i\theta} e^{\xi x - 1}|) \leq \sup_{\theta, \xi} g(e^{\operatorname{Re}(\xi x) - 1}) \leq g(e^{p(x) - 1})$. For every $x \in X$ the Hahn–Banach theorem gives a $\xi \in K$ with $\operatorname{Re}(\xi x) = p(x)$. On choosing $\theta \in [0, 2\pi]$ so that $e^{i\theta} e^{\xi x - 1} = e^{\operatorname{Re}(\xi x) - 1} = e^{p(x) - 1}$, we find that $\|h(x)\| = g(e^{p(x) - 1})$. Hence, $h \in \mathcal{O}(\Omega, Z)$, and $\|h(x)\| = 1/(1 - e^{p(x) - 1}) \rightarrow \infty$ as $x \in \Omega$ tends to point $x_0 \in X$ with $p(x_0) = 1$, in particular, to any boundary point $x_0 \in \partial\Omega$.

(b) Let $\omega = \{(x_1, \lambda) \in D \times \mathbb{C} : x_1 \in D, |\lambda| < R(x_1)\}$. As ω is pseudoconvex open in the complex Euclidean space $X_1 \times \mathbb{C}$, there is a proper holomorphic embedding $j: \omega \rightarrow \mathbb{C}^N$ for N high enough. Let K be the closed unit ball of the dual space X_2^* of X_2 endowed with the weak star topology, and for $(x_1, x_2) \in \Omega$ define $h(x_1, x_2) \in Z = C(K, \mathbb{C})$ (endowed with the sup norm) by $h(x_1, x_2)(\xi_2) = j(x_1, \xi_2 x_2)$ for $\xi_2 \in K$. Note that $\|h(x_1, x_2)\| = \sup_{|\lambda| \leq \|x_2\|} \|j(x_1, \lambda)\| \geq \|j(x_1, \|x_2\|)\|$ by the Hahn–Banach theorem, and the last tends to ∞ if $(x_1, \|x_2\|)$ tends to a boundary point of ω , in particular, when (x_1, x_2) tends in Ω to a boundary point of Ω in X .

(c) Let K be the closed unit ball of the dual space Z_1^* of Z_1 endowed with the weak star topology, and $Z = C(K, \mathbb{C})$ with the sup norm. For $x \in \Omega$ define $h(x) \in Z$ by $h(x)\zeta = g(\zeta f(x))$, where $\zeta \in K$ and $g(t) = 1/(1-t)$ for $t \in B_{\mathbb{C}}$ as in (a). Then $\|h(x)\| = g(\|f(x)\|)$ for $x \in \Omega$ by the Hahn–Banach theorem, and $h \in \mathcal{O}(\Omega, Z)$ is holomorphic. If $x \in \Omega$ tends to a boundary point $x_0 \in \partial\Omega$, then $x_0 \in \overline{\Omega} \subset \Omega'$, hence $x_0 \in \Omega'$ and $f(x) \rightarrow f(x_0)$, i.e., $\|f(x)\| \rightarrow \|f(x_0)\| = 1$, and $\|h(x)\| = 1/(1 - \|f(x)\|) \rightarrow \infty$.

The proof of Proposition 5.5 is complete.

6. THE PROOFS OF THEOREMS 1.3, 1.4, AND 1.5.

In this section we complete the proof of Theorems 1.3 on holomorphic domination, 1.4 on vanishing and Banach vector bundles, and 1.5 on the $\bar{\partial}$ -equation.

Proof of Theorem 1.3(a). Without loss of generality we may assume by Theorem 5.1 that X has a bimonotone Schauder basis. Let $u: X \rightarrow \mathbb{R}$ be the locally upper bounded function to be dominated. By paracompactness of X there is a continuous function $u_1: X \rightarrow [1, \infty)$ with $u(x) \leq u_1(x)$ for $x \in X$. Replacing u by u_1 , let us assume that $u \geq 1$ is continuous on X .

Suppose for a contradiction that u cannot be dominated by entire functions with values in Banach spaces on X . The hypothesis of Proposition 3.1 must then be false. Hence there is a ball $B_0 = B_X(x_0, 1)$ on which u cannot be dominated by entire functions with values in Banach spaces. The hypothesis of Proposition 4.2 must then also be false. So there is a ball $B_1 = B_X(x_1, 1/2)$

with $x_1 \in B_0$ such that u cannot be dominated by entire functions with values in Banach spaces on B_1 . Again, the hypothesis of Proposition 4.2 must be false and there is a ball $B_2 = B_X(x_2, 1/4)$ with $x_2 \in B_1$ such that u cannot be dominated by entire functions with values in Banach spaces on B_2 . Proceeding in this way we get a sequence of balls $B_n = B_X(x_n, 1/2^n)$ with $x_{n+1} \in B_n$ such that u cannot be dominated by entire functions with values in Banach spaces on B_n for $n \geq 0$.

As $x_{n+1} \in B_n$ we see that $\|x_{n+1} - x_n\| < 1/2^n$ and $\sum_{n=0}^{\infty} (x_{n+1} - x_n)$ is an absolutely convergent series in the Banach space X . Thus there is a limit $x_n \rightarrow x \in X$ as $n \rightarrow \infty$. Let $r > 0$ be so small that u is upper bounded on the ball $B_X(x, r)$. Choose $n \geq 0$ so large that $B_n \subset B_X(x, r)$. Hence u can be dominated by entire functions with values in Banach spaces on B_n after all, being upper bounded there. This contradiction completes the proof of (a).

Theorem 6.1. (a) *If M is as in Proposition 5.4, then holomorphic domination is possible in M .*

(b) *In particular, if X is a separable Banach space, and $\Omega \subset X$ open is as in Proposition 5.5, then holomorphic domination is possible in Ω .*

Proof. Part (a) follows from Theorem 1.3(a) via Proposition 5.3 upon embedding M in $C[0, 1]$ as a closed complex Banach submanifold. Part (b) follows from (a) by Proposition 5.5. The proof of Theorem 6.1 is complete, and as Theorem 1.3(b) is a special case of (b), the proof of Theorem 1.3 is also complete.

Theorem 6.2. *Let X be a Banach space with the bounded approximation property, $\Omega \subset X$ pseudoconvex open, and $S \rightarrow \Omega$ a cohesive sheaf. If plurisubharmonic domination is possible in Ω , then*

- (a) *the cohesive sheaf S admits a complete resolution over Ω , and*
- (b) *the sheaf cohomology group $H^q(\Omega, S)$ vanishes for all $q \geq 1$.*

Proof. Without loss of generality we may assume by Theorem 5.1 that X has a bimonotone Schauder basis. An inspection of the proof of the analogous Theorem 9.1 in [LP] reveals that therein it is enough to have plurisubharmonic domination in Ω and in those subsets of Ω to which Proposition 5.5 applies, and thus in which plurisubharmonic domination holds by Theorem 6.1. The proof of Theorem 6.2 is complete.

Proof of Theorem 1.4. Parts (a) and (b) follow directly from Theorem 6.2, (c) from [LP, §10] and Theorem 6.2, while (d) follows from [Pt2, Theorem 1.3(f)], completing the proof of Theorem 1.4.

Proof of Theorem 1.5. As the $\bar{\partial}$ -equation $\bar{\partial}u = f$ can be solved locally

on balls in Ω by a theorem of Defant and Zerhusen [DZ] (based upon the earlier work [L1] of Lempert) a standard step in one of the usual proofs of the Dolbeault isomorphism together with Theorem 1.4(c) completes the proof of Theorem 1.5.

Further applications of Theorems 1.4 and 1.5 can also be made, e.g., as in [DPV] or [LP].

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